Exercice 2 [TD2]

$$\dot{K} = sY - \delta K \qquad s \in [0,1], \ \delta \in [0,1]$$

$$Y = K^{\alpha}(AL)^{1-\alpha} \qquad \alpha \in [0,1]$$

$$\frac{\dot{L}}{L} = n > 0 \quad \forall t$$

(1) $$\frac{\dot{A}}{A} = x > 0 \quad \forall t$$

$\hookrightarrow$ $\dot{A}(t) = x\, A(t)$ EDL

Sous PT, avec $x = 0$, la croissance à LT des variables par tête est nulle $\neq$ observation

(2) Dynamique $\hat{k} = \frac{K}{AL}$

exprimer $\dot{\hat{k}}$ en fonction de $\hat{k}$ (niveau)

$$\dot{\hat{k}} = \frac{d}{dt}\left(\frac{K(t)}{A(t)L(t)}\right)$$

$$\dot{\hat{k}} = \frac{\dot{K}AL - K(\dot{A}L + A\dot{L})}{(AL)^2}$$

$$\dot{\hat{k}} = \frac{\dot{K}}{AL} - \underbrace{\frac{K}{AL}}_{\hat{k}} \cdot \frac{\dot{A}L + A\dot{L}}{AL} = \frac{\dot{K}}{AL} - \hat{k}(x+n)$$

$$\dot{\hat{k}} = \frac{sY - \delta K}{AL} - (n+x)\hat{k}$$

$$\Leftrightarrow \dot{\hat{k}} = s\hat{y} - (n+x+\delta)\hat{k}$$

$$Y = K^{\alpha}(AL)^{1-\alpha}$$

$$\hat{y} = \frac{Y}{AL}$$

$$\hat{y} = \frac{K^{\alpha}(AL)^{1-\alpha}}{AL}$$

$$\hat{y} = \frac{K^{\alpha}(AL)^{1-\alpha}}{(AL)^{\alpha}(AL)^{1-\alpha}}$$

$$a^{c} \cdot a^{b} = a^{c+b}$$

$$\hat{y} = \left(\frac{K}{AL}\right)^{\alpha}\left(\frac{AL}{AL}\right)^{1-\alpha} = \hat{k}^{\alpha}$$

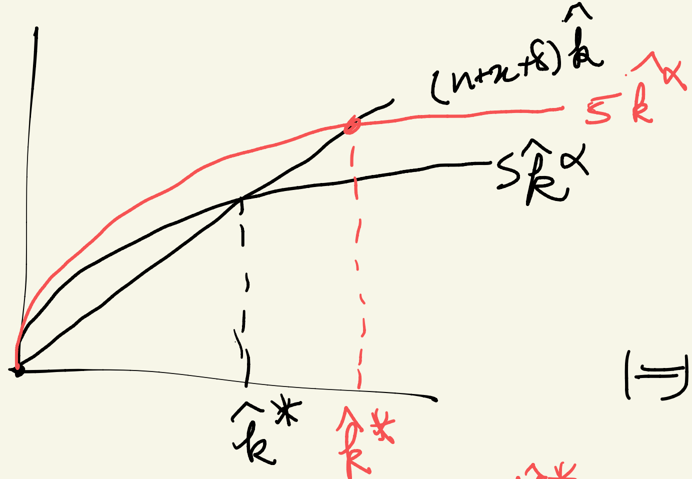
Au final : $$\dot{\hat{k}} = \underbrace{s\hat{k}^{\alpha}}_{\text{investissement par tête efficace}} - \underbrace{(n+x+\delta)\hat{k}}_{\text{dépréciation}}$$

(3) $\hat{k}^* > 0$ tq $\dot{\hat{k}} = 0$

$$\Updownarrow$$

$$s\hat{k}^{*\alpha} = (n+x+\delta)\hat{k}^*$$

$\exists!\ \hat{k}^* > 0$



$$\hat{k}^{*\,1-\alpha} = \frac{s}{n+x+\delta}$$

$$\iff \hat{k}^* = \left(\frac{s}{n+x+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$\frac{d\hat{k}^*}{ds} = \frac{1}{1-\alpha}\cdot\frac{1}{n+x+\delta}\left(\frac{s}{n+x+\delta}\right)^{\frac{\alpha}{1-\alpha}} > 0$$

Variables par tête

$$\dot{k} = \frac{\dot{K}}{L} - nk = \frac{sY - \delta K}{L} - nk$$

$$\dot{k} = sy - (n+\delta)k$$

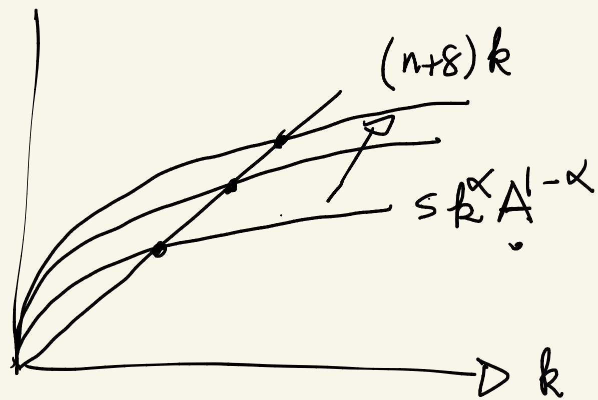
$$y = \frac{K^{\alpha}(AL)^{1-\alpha}}{L^{\alpha+1-\alpha}} = \left(\frac{K}{L}\right)^{\alpha} \cdot \left(\frac{AL}{L}\right)^{1-\alpha}$$

$$y = k^{\alpha} \cdot A^{1-\alpha}$$

$$\dot{k} = s k^{\alpha} A^{1-\alpha} - (n+\delta) k$$

croît à chaque instant $\frac{\dot{A}}{A} = x$

Cu



(4) $\quad \dot{\hat{k}} = s\hat{k}^{\alpha} - (n+x+\delta)\hat{k}$

Le taux de croissance

$$g_{\hat{k}} = \frac{\dot{\hat{k}}}{\hat{k}}$$

investissement brut

$$\Longleftrightarrow g_{\hat{k}} = \frac{s\hat{k}^{\alpha} - (n+x+\delta)\hat{k}}{\hat{k}}$$

— investissement net

$$\Longleftrightarrow \boxed{g_{\hat{k}} = s\hat{k}^{\alpha-1} - (n+x+\delta)}$$

$$\hat{k}^{*} = \left(\frac{s}{n+x+\delta}\right)^{\frac{1}{1-\alpha}}$$

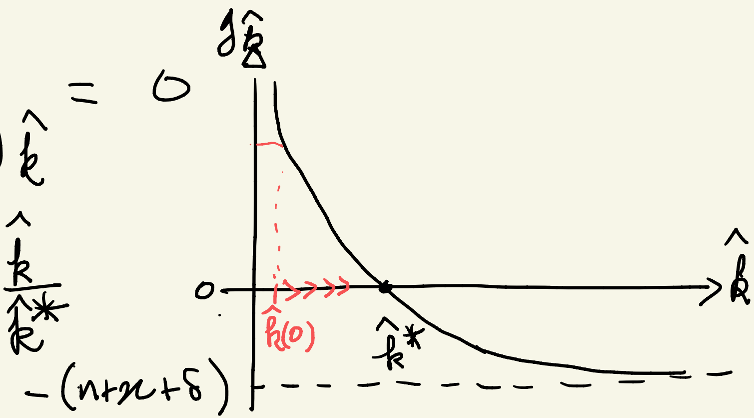
$$\hookrightarrow g_{\hat{k}} = (n+x+\delta)\left[\frac{s}{n+x+\delta}\hat{k}^{\alpha-1} - 1\right]$$

$$\Longleftrightarrow \quad g_{\hat{k}} = (n+x+\delta)\left[\hat{k}^{*\,1-\alpha}\, \hat{k}^{\alpha-1} - 1 \right]$$

$$\Longleftrightarrow \quad \boxed{g_{\hat{k}} = (n+x+\delta)\left[\left(\frac{\hat{k}}{\hat{k}^*}\right)^{\alpha-1} - 1 \right]}$$

Rmq si $\hat{k} = \hat{k}^*$ alors $g_{\hat{k}} = 0$

Δ relat° decroissante entre $g_{\hat{k}}$ et $\dfrac{\hat{k}}{\hat{k}^*}$



↳ $$\frac{dg_{\hat{k}}}{d\left(\hat{k}/\hat{k}^*\right)} = -(1-\alpha)(n+x+\delta)\left(\frac{\hat{k}}{\hat{k}^*}\right)^{\alpha-2} < 0$$

(5) $g_k = ?$

$$\hat{k} = \frac{K}{AL} = \frac{K/L}{A} = \frac{k}{A}$$

↳ $$A\hat{k} = k$$

$$\log A + \log \hat{k} = \log k$$

$$\frac{\dot{A}}{A} + \frac{\dot{\hat{k}}}{\hat{k}} = \frac{k}{k}$$

$$\Longleftrightarrow \boxed{g_k = g_{\hat{k}} + x}$$

De même $g_K = g_{\hat{k}} + x + n$

A LT $g_{\hat{k}} = 0 \Rightarrow g_k = x \quad g_K = x + n$

Dans la transition

$g_k > x$ ssi $\hat{k} < \hat{k}^*$

$$\hat{y} = \hat{k}^{\alpha}$$

$$\log \hat{y} = \alpha \log \hat{k}$$

en dérivant par t

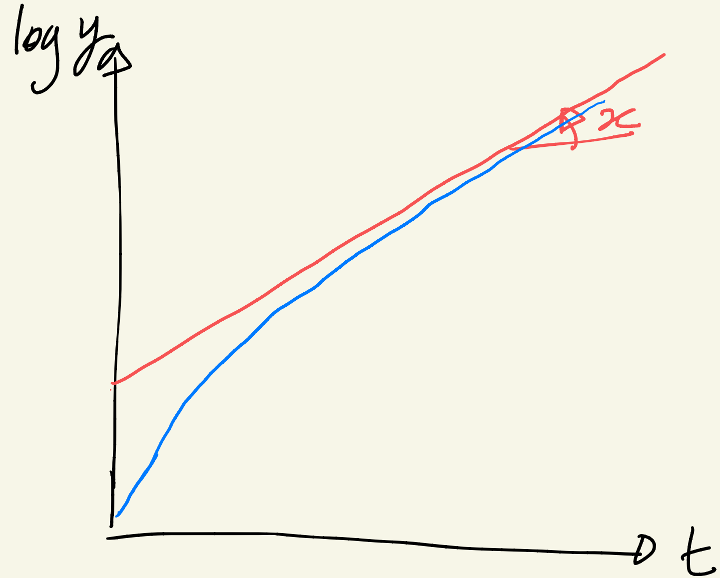
$$g_{\hat{y}} = \alpha \, g_{\hat{k}}$$

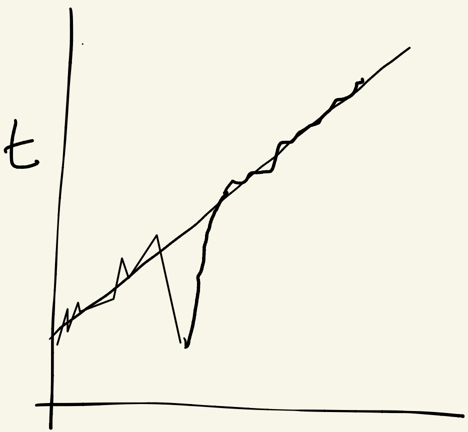
⚠ vrai seulement pour la fct Cobb Douglas

$$g_y = g_{\hat{y}} + x = \alpha g_{\hat{k}} + x$$

A LT $g_y = x$

Dans la transition $g_y > x$ ssi $\hat{k} < \hat{k}^*$


log y
t
x



[TD3] Exercice 1

$$\dot{K} = sY - \delta K$$

$$Y = K^{\alpha} L^{1-\alpha}$$

(1) $\dot{k} = \underbrace{sk^{\alpha}}_{i} - \underbrace{(n+\delta)k}_{\text{dépreciation}}$

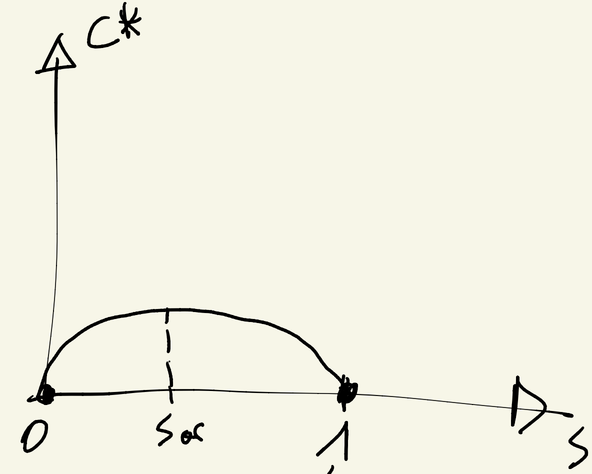
(2) $s{k^*}^{\alpha} = (n+\delta)k^*$

$\exists!\ k^* > 0$ solution

$$k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}} \qquad y^* = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

$$c^* = (1-s)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

(3) $s_{or} = \alpha$



$$c^*_{or} = (1-\alpha)\left(\frac{\alpha}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

(4) $\tau \in [0,1]$ appliqué à Y

$$\dot{k} = \underbrace{s\overbrace{(1-\tau)Y}^{\text{revenu disponible}}}_{\text{investissement privé}} + \underbrace{\tau Y}_{\text{investissement public}} - \delta k(t)$$

$$\dot{K} = \underbrace{\left(s(1-\tau) + \tau\right)}_{s_\tau = s + \tau(1-s)} Y - \delta K(t)$$

$$\text{LD} \quad \dot{k} = \left(s + \tau(1-s)\right) k(t)^{\alpha} - (n+\delta) k(t)$$

$$(S) \ \exists! \, k^* > 0 \ \text{tq} \quad \left(s + \tau(1-s)\right) k^{*\alpha} = (n+\delta) k^*$$

$$k^* = \left(\frac{s + \tau(1-s)}{n+\delta}\right)^{\frac{1}{1-\alpha}} \qquad y^* = \left(\frac{s+\tau(1-s)}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

revenu disponible à LT

$$c^* = (1-s)(1-\tau)\left[\frac{s+\tau(1-s)}{n+\delta}\right]^{\frac{\alpha}{1-\alpha}} \equiv c^*(\tau)$$

(b) $s < \alpha$ $\longrightarrow$ Sous accumulation

$$\bar{\tau} = \arg\max_{\{\tau\}} c^{*}(\tau)$$