Solow avec capital humain

2 facteurs accumulables :

- le capital physique K
- le capital humain H

$\Rightarrow \begin{cases} \text{épargne en capital physique} & s_K \\ \text{épargne en capital humain} & s_H \end{cases}$

avec $s_K > 0$, $s_H > 0$ et $s_K + s_H < 1$

$\hookrightarrow$ Il reste du revenu pour la consommation.

Fonction de production (Cobb_Douglas)

$$Y = K^{\alpha} . H^{\lambda} . [A(t)L(t)]^{1-\alpha-\lambda}$$

$\alpha \in]0,1[$, $\lambda \in]0,1[$

$\alpha + \lambda < \underline{1}$

Lois d'évolutions des stocks

$$\begin{cases} \overset{\circ}{K}(t) = s_K Y(t) - \delta K(t) \\ \overset{\circ}{H}(t) = s_H Y(t) - \delta . H(t) \end{cases}$$

$$\frac{\mathring{L}(t)}{L(t)} = n > 0 \quad \forall t$$

$$\frac{\mathring{A}(t)}{A(t)} = x > 0 \quad \forall t$$

On stationarise le modèle

$\longrightarrow$ variables par tête efficace

$$\hat{k}(t) = \frac{K(t)}{A(t)L(t)} \qquad \hat{h}(t) = \frac{H(t)}{A(t)L(t)}$$

$$\hat{y}(t) = \frac{Y(t)}{A(t)L(t)}.$$

On peut montrer que

$$\begin{cases} \dot{\hat{k}}(t) = s_K \hat{y}(t) - (n+x+\delta)\hat{k}(t) \\ \dot{\hat{h}}(t) = s_H \hat{y}(t) - (n+x+\delta)\hat{h}(t) \end{cases}$$

$$\hookrightarrow \begin{cases} \dot{\hat{k}}(t) = s_K \hat{k}(t)^{\alpha} \hat{h}(t)^{\lambda} - (n+x+\delta)\hat{k}(t) & (1) \\ \dot{\hat{h}}(t) = s_H \hat{k}(t)^{\alpha} \hat{h}(t)^{\lambda} - (n+x+\delta)\hat{h}(t) & (2) \end{cases}$$

Preuve de (1) :

$$\dot{\hat{k}} = \frac{d}{dt}\left(\frac{K}{AL}\right)$$

$$\Leftrightarrow \quad \mathring{\hat{k}} = \frac{\mathring{K} AL - K\left(\mathring{A}L + A\mathring{L}\right)}{(AL)^2}$$

$$\Leftrightarrow \quad \mathring{\hat{k}} = \frac{\mathring{K}}{AL} - \hat{k}\left(\frac{\mathring{A}\cancel{L}}{A\cancel{L}} + \frac{\cancel{A}\mathring{L}}{\cancel{A}L}\right)$$

$$\Leftrightarrow \quad \mathring{\hat{k}} = \frac{\mathring{K}}{AL} - (n+x)\hat{k}$$

$$\hookrightarrow \quad \mathring{\hat{k}} = \frac{s_K K^{\alpha} H^{\lambda} (AL)^{1-\alpha-\lambda} - \delta K}{AL} - (n+x)\hat{k}$$

$$\Leftrightarrow \quad \mathring{\hat{k}} = s_K \left(\frac{K}{AL}\right)^{\alpha} \left(\frac{H}{AL}\right)^{\lambda} \left(\frac{AL}{AL}\right)^{1-\alpha-\lambda} - (n+x+\delta)\hat{k}$$

$$\Leftrightarrow \quad \boxed{\dot{\hat{k}} = s_K \hat{k}^{\alpha} \hat{h}^{\lambda} - (n+x+\delta)\hat{k}}$$

cqfd

$$\begin{cases} \dot{\hat{k}} = s_K \hat{k}^{\alpha} \hat{h}^{\lambda} - (n+x+\delta)\hat{k} \\ \dot{\hat{h}} = s_H \hat{k}^{\alpha} \hat{h}^{\lambda} - (n+x+\delta)\hat{h} \end{cases}$$

Etat stationnaire $\quad \dot{\hat{k}} = \dot{\hat{h}} = 0$

$$\begin{cases} s_K \hat{k}^{*\alpha} \hat{h}^{*\lambda} = (n+x+\delta)\hat{k}^{*} \\ s_H \hat{k}^{*\alpha} \hat{h}^{*\lambda} = (n+x+\delta)\hat{h}^{*} \end{cases}$$

En faisant le rapport des 2 dernières équations

$$\frac{\hat{k}^*}{\hat{h}^*} = \frac{s_K}{s_H}$$

$$\iff \quad \hat{k}^* = \frac{s_K}{s_H} \hat{h}^*$$

$$s_H \, \hat{k}^{*\alpha} \, \hat{h}^{*\lambda} = (n+x+\delta) \, \hat{h}^*$$

$$\hookrightarrow \quad s_H \left(\frac{s_K}{s_H}\right)^{\alpha} \hat{h}^{*\alpha} \, \hat{h}^{*\lambda} = (n+x+\delta) \, \hat{h}^*$$

$$\iff \quad s_K^{\alpha} s_H^{1-\alpha} \, \hat{h}^{*\alpha+\lambda} = (n+x+\delta) \, \hat{h}^*$$

$$\hat{h}^{*\,1-\alpha-\lambda} = \frac{s_K^{\alpha} s_H^{1-\alpha}}{n+x+\delta}$$

$$\boxed{\hat{h}^* = \left(\frac{s_K^{\alpha} \cdot s_H^{1-\alpha}}{n+x+\delta} \right)^{\frac{1}{1-\alpha-\lambda}}}$$

De même

$$\boxed{\hat{k}^* = \left(\frac{s_K^{1-\lambda} \, s_H^{\lambda}}{n+x+\delta} \right)^{\frac{1}{1-\alpha-\lambda}}}$$

$$\hat{y}^* = \hat{k}^{*\alpha} \, \hat{h}^{*\lambda} \qquad \frac{\lambda}{1-\alpha-\lambda} \qquad \frac{\alpha}{1-\alpha-\lambda}$$

$$\hat{y}^* = \left(\frac{s_K^{\alpha} \, s_H^{1-\alpha}}{n+x+\delta} \right)^{\frac{\lambda}{1-\alpha-\lambda}} \cdot \left(\frac{s_K^{1-\lambda} \, s_H^{\lambda}}{n+x+\delta} \right)^{\frac{\alpha}{1-\alpha-\lambda}}$$

$$\hat{y}^* = (n+x+\delta)^{-\frac{\alpha+\lambda}{1-\alpha-\lambda}} \cdot s_K^{\frac{\alpha(1-\lambda)+\lambda\alpha}{1-\alpha-\lambda}} \cdot s_H^{\frac{\alpha\lambda+(1-\alpha)\lambda}{1-\alpha-\lambda}}$$

$$\Leftrightarrow \boxed{\hat{y}^* = (n+x+\delta)^{-\frac{\alpha+\lambda}{1-\alpha-\lambda}} \cdot s_K^{\underbrace{\frac{\alpha}{1-\alpha-\lambda}}} \cdot s_H^{\underbrace{\frac{\lambda}{1-\alpha-\lambda}}}}$$

élasticité de $\hat{y}^*$ par rapport au taux d'épargne en capital physique

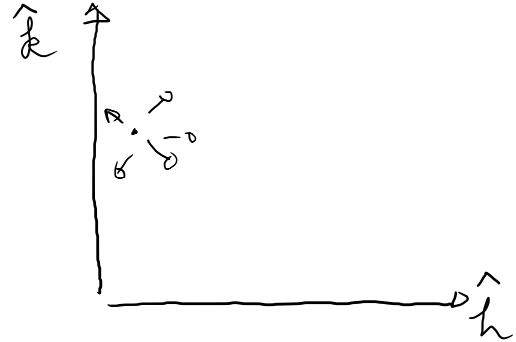
élasticité $\hat{y}^* / s_H$

$$\frac{\alpha}{1-\alpha} < \frac{\alpha}{1-\alpha-\lambda}$$

introduire H augmente l'élasticité de $\hat{y}^*$ / épargne en capital physique

$$\begin{cases} \dot{\hat{k}} = s_K \hat{k}^{\alpha} \hat{h}^{\lambda} - (n+x+\delta)\hat{k} & (1) \\ \dot{\hat{h}} = s_H \hat{k}^{\alpha} \hat{h}^{\lambda} - (n+x+\delta)\hat{h} & (2) \end{cases}$$

diagramme des phases



$$\dot{\hat{k}} > 0 \iff s_K \hat{k}^{\alpha} \hat{h}^{\lambda} - (n+x+\delta)\hat{k} > 0$$

$$\iff s_K \hat{h}^{\lambda} - (n+x+\delta)\hat{k}^{1-\alpha} > 0$$

$$\iff (n+x+\delta)\hat{k}^{1-\alpha} < s_K \hat{h}^{\lambda}$$

$$\iff \hat{k}^{1-\alpha} < \frac{s_K}{n+x+\delta} \cdot \hat{h}^{\lambda}$$

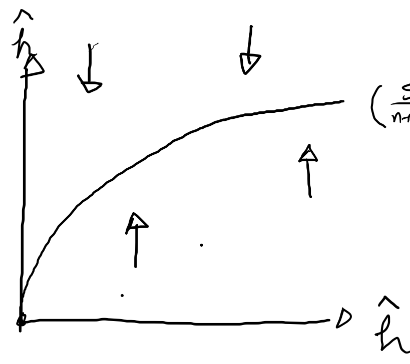
$$\iff \hat{k} < \left(\frac{s_K}{n+x+\delta}\right)^{\frac{1}{1-\alpha}} \cdot \hat{h}^{\frac{\lambda}{1-\alpha}} \quad < 1$$

$$\frac{\lambda}{1-\alpha} < 1$$

$$\iff \lambda < 1-\alpha$$

$$\iff \lambda + \alpha < 1$$

vrai par hypothèse



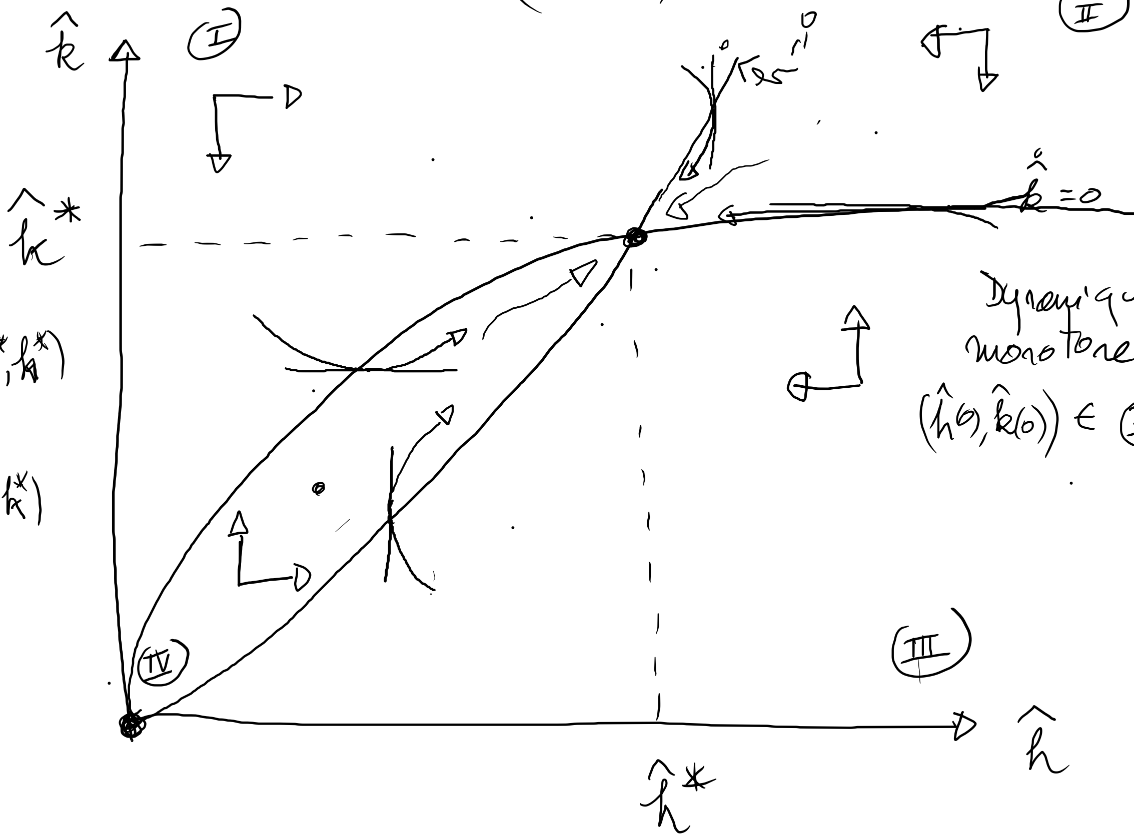
De la même façon on montre que

$$\mathring{\hat{h}} > 0 \iff \hat{k} > \left(\frac{n+r+\delta}{s_H}\right)^{\frac{1}{\alpha}} \cdot \hat{h}^{\frac{1-\lambda}{\alpha}} \quad > 1$$

$\frac{1-\lambda}{\alpha} > 1$

$\iff 1-\lambda > \alpha$

$\iff \alpha + \lambda < 1$



(II) → (IV) } → $(\hat{k}^*, \hat{h}^*)$
(III) → (IV)

(I) → (II) } → $(\hat{h}^*, \hat{k}^*)$
(III) → (II)

Dynamique monotone ssi $(\hat{h}(0), \hat{k}(0)) \in$ (II) ou (IV)

Modèle de Solow + Capital humain dans le plan

$$\left(\log y(0) \; , \; g_y \right)$$

- Approximation dans un voisinage de $(\hat{k}^*, \hat{h}^*)$

$$z(t) = z(0)\, e^{-\beta t}$$

$$\log \frac{\hat{y}(t)}{\hat{y}^*} \qquad \beta = (1 - \alpha - \lambda)(n + x + \delta) > 0$$

- Le taux de croissance moyen de la product° par tête

$$g_{y,i} = x + \frac{1 - e^{-\beta t}}{t} a + \frac{1 - e^{-\beta t}}{t} \log \hat{y}_i^* - \frac{1 - e^{-\beta t}}{t} \log y(0) + \frac{1 - e^{-\beta t}}{t} \epsilon_i$$

$$\log \hat{y}^*_i = -\frac{\alpha+\beta}{1-\alpha-\beta}\log(n_i+x+\delta) + \frac{\alpha}{1-\alpha-\beta}\log s_{K,i} + \frac{\lambda}{1-\alpha-\lambda}\log s_{H,i}$$

$$\beta = (1-\alpha-\lambda)(n+x+\delta)$$

On a donc

$$g_{y,i} = x + \frac{1-e^{-\beta t}}{t}a + \frac{\alpha}{1-\alpha-\lambda}\frac{1-e^{-\beta t}}{t}\log s_{K,i} + \overbrace{\frac{\lambda}{1-\alpha-\lambda}\frac{1-e^{-\beta t}}{t}\log s_{H,i}}$$
$$- \frac{\alpha+\lambda}{1-\alpha-\lambda}\cdot\frac{1-e^{-\beta t}}{t}\log(n_i+x+\delta)$$
$$- \frac{1-e^{-\beta t}}{t}\log y_i(0) + u_i$$

Modèle empirique :

$$g_{y,i} = c_0 + c_1\log \overset{I/Y}{s_{K,i}} + c_2\log s_{H,i} + c_3\log(\overset{PNT}{n_i} + \overset{4\%}{x+\delta}) + c_4\log y_i(0) + u_i$$

$$c_1 > 0,\quad c_2 > 0,\quad c_3 < 0,\quad c_4 < 0$$

et $c_1 + c_2 + c_3 = 0$

Modèle empirique :

$$g_{y,i} = c_0 + c_1 \log s_{k,i} + c_2 \log SCHOOL_i + c_3 \log(n_i + x + \delta) + c_4 \log y_i(0) + u_i$$

$$c_1 > 0, \quad c_2 > 0, \quad c_3 < 0, \quad c_4 < 0$$

$$c_1 + c_2 + c_3 = 0$$

TESTS FOR CONDITIONAL CONVERGENCE

Dependent variable: log difference GDP per working-age person 1960–1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	3.04	3.69	2.81
	(0.83)	(0.91)	(1.19)
ln(Y60)	−0.289	−0.366	−0.398
	(0.062)	(0.067)	(0.070)
ln(I/GDP)	0.524	0.538	0.335
	(0.087)	(0.102)	(0.174)
$\ln(n + g + \delta)$	−0.505	−0.551	−0.844
	(0.288)	(0.288)	(0.334)
ln(SCHOOL)	0.233	0.271	0.223
	(0.060)	(0.081)	(0.144)
$\bar{R}^2$	0.46	0.43	0.65
s.e.e.	0.33	0.30	0.15
Implied λ	0.0137	0.0182	0.0203
	(0.0019)	(0.0020)	(0.0020)

Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.

$$\beta = \underbrace{\left(1 - \underset{1/3}{\alpha} - \underset{1/3}{\lambda}\right)}_{1/3} \underbrace{\left(\underset{2\%}{n} + \underset{2\%}{x} + \underset{2\%}{\delta}\right)}_{6\%} = 2\%$$